Islands and closed universes

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Island formula applied to closed universes. The island formula has an interesting application to closed gravitating universes [1]. By a closed universe, we mean (a possibly disconnected) spacetime manifold with compact spatial slices. Imagine a semiclassical setting where the closed universe contains a few qubits that are entangled with partner qubits in an auxiliary, non-gravitating system, see figure 12b of [1].

What is the entropy of the qubits in the auxiliary system? The naive answer is $k \log 2$, where k is the number of qubits under consideration. However, since there is gravity in the closed universe, we should allow for the entanglement wedge of these qubits to contain a part or all of the closed universe. If we include all of the closed universe in the entanglement wedge, we get zero for the generalized entropy since the island region has no boundary, so we don't pay any area cost (notice this is trivially an extremum since there is no boundary to vary). We have also included all the partner qubits so S_{bulk} is zero. Altogether, we conclude that the generalized entropy is actually zero [1].

If we think about the dimensionality of the Hilbert space of the closed universe system, this result is a bit confusing. See for example [2]. Imagine that the Hilbert space of the closed universe system contained at least two orthogonal states $|a\rangle_{\text{CU}}$ and $|b\rangle_{\text{CU}}$. This is allowed in explicit models like those in [3]; for instance the states $|a\rangle_{\text{CU}}$ and $|b\rangle_{\text{CU}}$ could be two of the infinitely many alpha states. Then we can take an auxiliary qubit and consider the state $\frac{1}{\sqrt{2}}\left(|a\rangle_{\text{CU}}|0\rangle_{\text{aux}} + |b\rangle_{\text{CU}}|1\rangle_{\text{aux}}\right)$. Clearly, by explicitly taking the partial trace we see that the von Neumann entropy of the auxiliary system is $\log 2$, and not zero, which conflicts with the argument of [1].[1] The states $|a\rangle_{\text{CU}}$ and $|b\rangle_{\text{CU}}$ can be chosen such that they have a simple path integral preparation, and in e.g. the model of [3] this nonzero entropy is explicitly reproduced by path integrals, as it must be since the path integrals in that model are computing quantum-mechanical amplitudes. In particular the state $|a\rangle_{\text{CU}}$ can have a semiclassical realization as a universe with a single connected component and a qubit with spin up, while state $|b\rangle_{\text{CU}}$ has the same semiclassical realization except the qubit has spin down.

In this brief note, we point out that the resolution of the above puzzle is simply that the island formula $S(R) = \min \text{ext}_I \left(\text{Area}(\partial I)/4 + S_{\text{mat}}(R \cup I)\right)$ computes $\mathbb{E}[S(\rho)]$, where the expectation value is taken in the ensemble defined by the Hartle-Hawking state of the closed universe sector (and ρ is the density matrix of the auxiliary non-gravitating system R); explicitly $\mathbb{E}[S(\rho)] = \langle \text{HH}|\widehat{S(\rho)}|\text{HH}\rangle$.[2] This is implied by [3], and was stated explicitly in section 4.2 of [5], but we think that it will be useful to spell out a few more details clearly in this note. If the closed

[1] An alternative is to claim that the Hilbert space of the closed universe system is one-dimensional, see for example [4] which states this condition as a swampland conjecture, although the idea is an older one.

[2] In the notation introduced in [3], all boundary observables become operators in the bulk Hilbert space.

universe sector is in a different state, the ensemble probabilities will change; in this case we can set up the boundary replica trick by imposing appropriate boundary conditions on the gravitational path integral, and it will still compute $\mathbb{E}[S(\rho)]$ in the new ensemble. An extreme special case is when the closed universe is in a particular alpha state $|\alpha\rangle$, which represents a delta-function peaked ensemble. In this case $\mathbb{E}[S(\rho)] = \langle\alpha|\,\widehat{S(\rho)}\,|\alpha\rangle = S(\rho_\alpha)$.

Consider a state of the form

$$|\Psi\rangle = \sum_{i=1}^{N} \sqrt{p_i}\; |\alpha_i\rangle_{\text{CU}}\, |\psi_i\rangle_{\text{aux}} \tag{1}$$

where the states $|\psi_i\rangle$ belong to the Hilbert space of an auxiliary, non-gravitating system which could be, for example, the Hawking radiation that emanated from a black hole that is now fully evaporated. The size and the precise initial state of the black hole is encoded in the nature of the state $|\Psi\rangle$, for instance in the parameter N. In the island (or the gravitational path integral with HH boundary conditions) computation of the entropy of the auxiliary system, one computes

$$\mathbb{E}[S(\rho_{\text{aux}})] = \sum_{i=1}^{N} p_i\, S(|\psi_i\rangle\langle\psi_i|) = \sum_{i=1}^{N} (p_i \times 0) = 0\,. \tag{2}$$

Note that this is not the same as computing the entropy of the aux system in the state (1), since

$$S(\text{Tr}_{\text{CU}}\,|\Psi\rangle\langle\Psi|) = \sum_{i=1}^{N} -p_i \log p_i \neq 0\,. \tag{3}$$

Thus, the island formula does not capture the entropy due to correlations with the closed-universe sector.

View from the SYK model. The process of black hole evaporation in the bulk can be stated in the SYK (+wire) model as the preparation of the same initial state in all members of the SYK ensemble. For example, we could form spin operators by pairing up the Majorana fermions and taking the state to be all spins in the $+z$ eigenstate. Even though the initial state in every member of the ensemble is the same, the different members evolve by a different Hamiltonian. Translating the ensemble into ordinary quantum mechanics with superselection sectors, we have:

$$\sum_{J_{ijkl}} |+\ldots+\rangle_{\text{SYK}} \otimes |J_{ijkl}\rangle \to \sum_{J_{ijkl}} e^{-\mathrm{i}tH[J_{ijkl}]}\, |+\ldots+\rangle_{\text{SYK}}\, |J_{ijkl}\rangle \tag{4}$$

The initial state is a product state but evolves into a bipartite entangled state, thought of as an ensemble of pure states. The quantity $\mathbb{E}[S(\rho_{\text{SYK}})]$ is clearly zero, since all final states in the ensemble are pure. This is reproduced by the island formula.

In contrast, the state (1) or (4) elevates the set of coupling constants to a factor in the Hilbert space, which is identified with the closed universe factor in the dual gravitational language. Importantly, the operator algebra on this factor is abelian,[3] since the coupling constant sector has no

[3] This means that all operators are block diagonal, with the blocks being labeled by the coupling constants or the alphas. In the SYK setting, this is manifest, since although the J_{ijkl} variables are integrated over in the disorder-averaged path integral, they do not have kinetic terms. On the gravitational side, the abelian property of the operator algebra is not automatic and is a choice that is made in the definition of the theory [5,6].

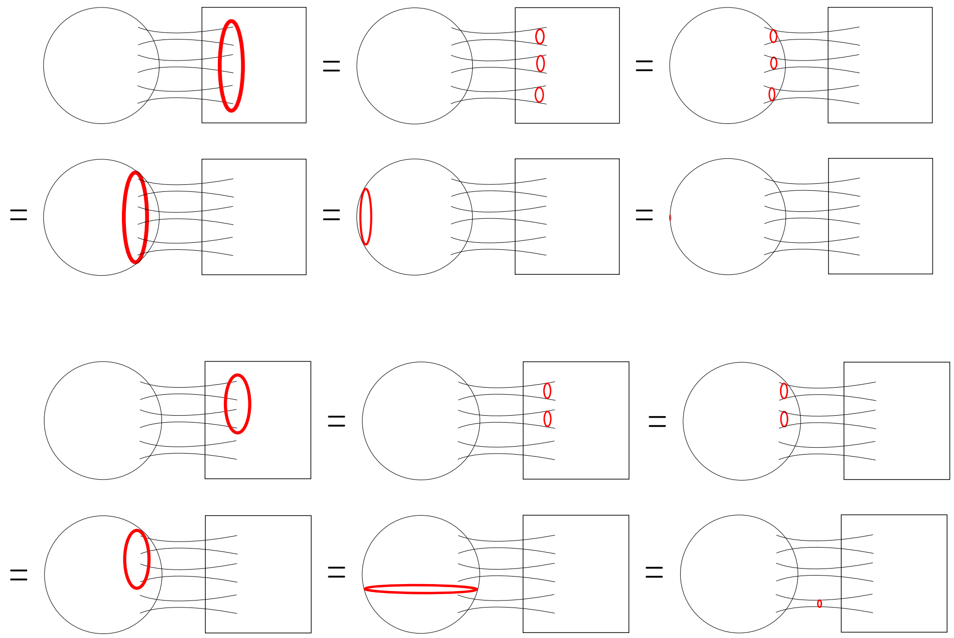

Figure 1: Entangling three qubits in a closed universe into Bell pairs with auxiliary qubits in an open universe. The entropy of all three qubits in the auxiliary universe vanishes since the surface can travel to the sphere and slip off (first two lines), but the entropy of any subset is nonvanishing and equals the entropy of the complement (final two lines).

dynamics. If we simply trace out the coupling constant sector in (4), we will get a nonzero entropy for the SYK factor. This is precisely the entropy captured in (3).

We can divide the SYK state into two pieces to model the bulk problem of considering a subset of the Hawking radiation. The Page curve is reproduced as one considers subsets ranging from the empty set to the entire set. Any proper subset is then thought of as in a tripartite state with the rest of the SYK state and the closed universe sector, and its entanglement cannot be unambiguously ascribed to one or the other party.

ER=EPR for subsets of closed universes. How does ER = EPR work for closed universes? Nothing stops us from considering geometries with wormholes appended to our closed universe, see figure 1.[4] Thinking in terms of homologous minimal surfaces in the geometry of figure 1, we get a zero answer for the entanglement entropy of the *entire* auxiliary system.[5] This seems to say that there is no entanglement. However, the main point is that, just as we discussed in the SYK setting, *subsystems* of the auxiliary system can still have nontrivial entanglement, even if the closed universe sector is in an alpha state.

In figure 1, we have drawn the case of three qubits in the closed universe, each in a Bell pair with a partner qubit in an auxiliary non-gravitating system. If we compute the entropy of all three qubits in the auxiliary system, we see that the minimal surface can slide across the bridges to the closed universe and contract to vanishing size, giving zero entropy.[6]

[4] See [7, 8] for similar toy examples in the context of black hole evaporation.

[5] Throughout this section, when we use the word entropy, we will be talking about $\mathbb{E}[S(\rho)]$, with the ensemble defined by the Hartle-Hawking state of the closed universe.

[6] To avoid the confusions of quantum wormholes, we can think of the qubits instead as black holes and place them

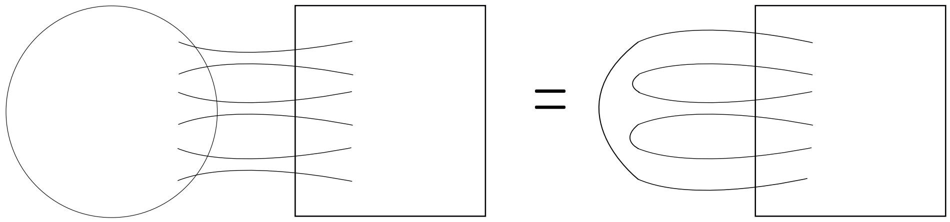

Figure 2: Attempting to entangle yourself with a closed universe leads to entangling yourself with yourself.

However, a subset of qubits in the auxiliary universe can give a nonvanishing entropy: if we compute the entropy of one of them, there is an obstruction to the minimal surface sliding off, and we will get $\log 2$ from the minimal surface sitting at the waist of the wormhole. If we compute the entropy of the other two, there is a minimal surface that is two circles sitting at the waists of the two wormholes, but this can be further minimized and is homologous to a single circle sitting at the waist of the third wormhole. This latter surface gives again $\log 2$. This is as expected, since the global state was pure, so the entropy of a region should equal the entropy of its complement. Physically this entanglement should be interpreted as being between *subsets* of the auxiliary universe, and not with the closed universe itself. This can be suggestively illustrated as in figure 2, and is also a cartoon of the nature of Hawking radiation once a black hole has completely evaporated.

References

[1] Ahmed Almheiri, Raghu Mahajan, Juan Maldacena, and Ying Zhao, "The Page curve of Hawking radiation from semiclassical geometry," JHEP **03**, 149 (2020), arXiv:1908.10996 [hep-th]

[2] Vijay Balasubramanian, Arjun Kar, and Tomonori Ugajin, "Islands in de Sitter space," JHEP **02**, 072 (2021), arXiv:2008.05275 [hep-th]

[3] Donald Marolf and Henry Maxfield, "Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information," JHEP **08**, 044 (2020), arXiv:2002.08950 [hep-th]

[4] Jacob McNamara and Cumrun Vafa, "Baby Universes, Holography, and the Swampland," (4 2020), arXiv:2004.06738 [hep-th]

[5] Tarek Anous, Jorrit Kruthoff, and Raghu Mahajan, "Density matrices in quantum gravity," SciPost Phys. **9**, 045 (2020), arXiv:2006.17000 [hep-th]

[6] Eduardo Casali, Donald Marolf, Henry Maxfield, and Mukund Rangamani, "Baby Universes and Worldline Field Theories," (1 2021), arXiv:2101.12221 [hep-th]

[7] Leonard Susskind, "ER=EPR, GHZ, and the consistency of quantum measurements," Fortsch. Phys. **64**, 72–83 (2016), arXiv:1412.8483 [hep-th]

[8] Chris Akers, Netta Engelhardt, and Daniel Harlow, "Simple holographic models of black hole evaporation," JHEP **08**, 032 (2020), arXiv:1910.00972 [hep-th]

in thermofield double states with black holes in the auxiliary system.