The typical nature of the divergence may easily be understood on a simple example. For this purpose we replace path integrals by ordinary ones such as

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\varphi \, e^{-(\varphi^2/2 + g\varphi^4)} \tag{9-177}$$

Of course, this case is trivial enough that we might obtain $Z(g)$ in closed form. The point $g = 0$ is an essential singularity. When $\operatorname{Re} g$ becomes negative (and $|\operatorname{Arg} g| < \pi$) we may still rotate the integration contour in (9-177). This is no longer possible when g approaches the negative real axis and the integral blows up for large φ. We may think of φ as the value of a field at a point with $\varphi^2/2 + g\varphi^4$ as a caricature of the action. Negative values of g correspond to an unstable situation where the "potential" is not bounded from below. This is reflected in the perturbative expansion if we write

$$Z(g) = \sum_0^\infty g^k Z_k$$

$$Z_k = \frac{(-1)^k}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\varphi \, \frac{\varphi^{4k}}{k!} \, e^{-\varphi^2/2} = (-1)^k 4^k \, \frac{\Gamma(2k + \frac{1}{2})}{\sqrt{\pi} k!} \tag{9-178}$$

Using Stirling's formula

$$k! \sim \sqrt{2\pi k} \, e^{k \ln k - k}$$

for large k, we find that Z_k behaves as

$$Z_k \sim \frac{(-16)^k}{\sqrt{\pi}} \, e^{(k - 1/2)\ln k - k} \tag{9-179}$$

Nevertheless, the power series in g is asymptotic in the complex g plane cut along the negative real axis since

$$\left| Z(g) - \sum_0^n Z_k g^k \right| < \frac{4^{n+1}\Gamma(2n + \frac{3}{2})}{\sqrt{\pi}(n + 1)!} \, \frac{|g|^{n+1}}{[\cos(\frac{1}{2} \operatorname{Arg} g)]^{2n + 3/2}} \tag{9-180}$$

meaning that for fixed n and g small enough the right-hand side may be made arbitrarily small.

The asymptotic behavior of Z_k was obtained by applying Stirling's formula to the exact expression. The latter will not be available in more realistic cases. However, this suggests to apply the method of steepest descent for large k to

$$Z_k = \frac{(-1)^k}{k!}\sqrt{\frac{2}{\pi}}\int_0^\infty d\varphi\, e^{-\varphi^2/2 + 4k\ln\varphi} \tag{9-181}$$

The position of the saddle point is

$$\varphi_c^2 = 4k \tag{9-182}$$

and integration over quadratic deviations from φ_c yields

$$Z_k \sim \frac{(-1)^k}{k!}\sqrt{2}\, e^{2k\ln 4k - 2k} = \frac{(-16)^k}{\sqrt{\pi}}\, e^{(k-1/2)\ln k - k} \tag{9-183}$$

as before.

What can we do from such a divergent series besides using it in an asymptotic sense to evaluate the function for small g? In some fortunate circumstances, such as the one discussed here, the power series (9-178) contains in fact enough information to reconstruct the function unambiguously. Of course, it might be argued that we could add any function such as $\exp(-1/\sqrt{g})$ with vanishing derivatives at the origin along the real positive axis. However, within a well-defined class of functions excluding these pathologies [and to which $Z(g)$ belongs] a Borel transformation enables us to recover $Z(g)$ from its divergent perturbative series.

Introduce

$$B(t) = \sum_0^\infty \frac{Z_k t^k}{\Gamma(k+\frac{3}{2})} \tag{9-184}$$

where the choice of $\Gamma(k+\frac{3}{2})$ is justified by the growth given in Eq. (9-179) and could be slightly modified in analogous cases. The series on the right-hand side in (9-184) will converge within a circle of finite radius in the complex t plane. If $B(t)$ may be continued along the entire real positive axis and does not grow too fast at infinity, then $Z(g)$ will be given by

$$Z(g) = \int_0^\infty dt\, t^{1/2} B(gt)\, e^{-t} \tag{9-185}$$

Knowledge of the perturbative series is insufficient to prove Borel summability. It is, however, sufficient to disprove it in the case where $B(t)$ would be found to have a singularity on the positive real axis. This happens, for instance, if asymptotically the Z_k have equal phases.

For our simple example we find

$$B(t) = \frac{2}{\sqrt{\pi}}\sqrt{1-u}$$
$$u = \frac{\sqrt{1+16t}-1}{\sqrt{1+16t}+1} \tag{9-186}$$

and $B(t)$ is analytic in a plane cut from $-\frac{1}{16}$ to $-\infty$.

To obtain a convergent expansion for $Z(g)$, let us map the cut t plane onto a circle, keeping the origin fixed (this is given here by the choice of variable u), and derive the convergent Taylor series for $\tilde{B}(u)$:

$$B(t) = \tilde{B}(u) = \sum_{0}^{\infty} b_k [u(t)]^k \tag{9-187}$$

from the knowledge of its expansion in t. Then

$$Z(g) = \sum_{0}^{\infty} b_k \int_{0}^{\infty} dt\, t^{1/2} [u(gt)]^k e^{-t} \tag{9-188}$$

Since b_k decreases here as $k^{-3/2}$ it is easy to see that this new series will converge as $e^{-3(k/3g^{1/2})^{2/3}}$.